

A formal proof of the optimal frame setting for Dynamic-Frame Aloha with known population size

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Abstract

In Dynamic-Frame Aloha subsequent frame lengths must be optimally chosen to maximize throughput. When the initial population size \mathcal{N} is known, numerical evaluations show that the maximum efficiency is achieved by setting the frame length equal to the backlog size at each subsequent frame; however, at best of our knowledge, a formal proof of this result is still missing, and is provided here. As byproduct, we also prove that the asymptotical efficiency in the optimal case is e^{-1} , provide upper and lower bounds for the length of the entire transmission period and show that its asymptotical behaviour is $\sim ne - \zeta \ln(n)$, with $\zeta = 0.5/\ln(1 - e^{-1})$.

Index Terms

RFID, Collision Resolution, Frame Aloha, Frame Length, Optimal Strategy.

I. INTRODUCTION

Collision resolution protocols have played a fundamental role in communication systems starting with the appearance of the Aloha protocol [1], [2] back in 1970. Since then, a variety of such protocols have appeared and have influenced Satellite, Radio and Local Area Networks communications, being nowadays applied also to Radio Frequency Identification (RFID) systems [3],[4]. In RFID systems a reader interrogates a set of tags in order to identify each of them [4]. Collisions may occur among the responses of tags and collision resolution protocols are used to arbitrate collisions so that all tags can be finally identified. In this environment, the number of tag to be identified \mathcal{N} is not a random variable as it happens in multiple access systems, but is rather a constant n either known, or unknown; nevertheless the collision resolution problem is quite similar in both environments and RFID protocols are a straightforward derivation of those proposed for multiple access.

Among the different protocols envisaged in past years, Dynamic Frame Aloha (DF-Aloha) is the most popular in RFID [5], [6]. In Frame-Aloha (F-Aloha) time is divided into time slots equal to a packet transmission time,

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slots are grouped into frames, and a tag is allowed to transmit only a single packet per frame in a randomly chosen slot. In the first frame all tags transmit, but only a part of them avoid collisions with other transmissions and get through. The remaining ones, referred to as the backlog, re-transmit in the subsequent frames until all of them are successful.

Unfortunately F-Aloha, like other protocols of the Aloha family, is intrinsically unstable and its throughput is very small unless some stabilizing control is used. A way to do this is to dynamically adapt the frame length r according to the backlog size n , hence the name Dynamic Frame Aloha (DF-Aloha). This strategy has been proposed for the first time in [7], in the field of satellite communications, where the author proposes to set the frame length exactly equal to a backlog estimate \hat{n} . The reason for adopting this strategy is that the throughput in a slot of a frame of length r is

$$\left(1 - \frac{1}{r}\right)^{n-1} \quad (1)$$

and is thus maximized for $r = n$.

As a matter of fact, the performance figure to be optimized in choosing the size of each frame is the overall efficiency

$$\eta = \frac{E[\mathcal{N}]}{E[\mathcal{L}]} \quad (2)$$

where \mathcal{N} is the original tag population size and \mathcal{L} is the number of slots needed to successfully transmit all the backlog, the latter identified as identification period (IP). In RFID systems \mathcal{N} is usually a constant n and, therefore, the efficiency is maximized by minimizing $E[\mathcal{L}(n)]$.

In [7] a recursive formula for the calculation of $E[\mathcal{L}(n)]$, reported later in this paper, is given, and, when applied to known n , numerically shows that the strategy that sets $r = n$ at each frame provides the shortest $E[\mathcal{L}(n)]$ for any value of n attempted. However, up to now, to the knowledge of authors, none has provided a theoretical verification of the result.

Subsequent papers dealing with DF-Aloha, even the more recent dealing with RFID, (see for example [8][9][3]), have often assumed $r = \hat{n}$ as optimal choice, never really discussing the optimal strategy, with the notable exception of [10]. In this paper the authors have pointed out the non-optimality of the above setting, and suggest a procedure to numerically find the best frame-length choice when the initial backlog size n is known in distribution. This procedure, when applied to known n , provides the recursive formula cited above for $E[\mathcal{L}(n)]$, which is still solved only numerically.

In this paper we present an analysis of DF-Aloha with known backlog size n and complete exploration of the frame, that definitely proves that local optimization, i.e., maximizing the throughput/efficiency in each frame ($r = n$) also maximizes the overall efficiency. We also rigorously prove that the asymptotical efficiency in the optimal case is e^{-1} , provide upper and lower bounds for $E[\mathcal{L}(n)]$ and show that its asymptotical behaviour is $\sim ne - \gamma \ln(n)$, with $\gamma = 0.5 / \ln(1 - e^{-1})$. The basic demonstrations are given in the next section, while Lemmas are grouped in the Appendix.

II. ANALYSIS

Let n be the number of terminals to be identified and $L(n, r_n) = E[\mathcal{L}(n)]$ the average length of the IP, being r_n the length of the first frame. The efficiency of the procedure is defined as

$$\eta(n) = \frac{n}{L(n, r_n)} \quad (3)$$

where $L(n, r_n)$ can be expressed as [7]

$$L(n, r_n) = r_n + \sum_{s=0}^m p_{n, r_n}(s) L(n-s, r_{n-s}), \quad n \geq 2, \quad (4)$$

being $m = \min\{n-2, r_n-1\}$, and $\{p_{n, r_n}(s)\}_s = \{P(\mathcal{S}_{n, r_n} = s)\}_s$ the distribution of the number of successes \mathcal{S}_{n, r_n} in the first frame, of length r_n . Expliciting term $L(n, r_n)$ yields

$$L(n, r_n) = \frac{r_n + \sum_{s=1}^m p_{n, r_n}(s) L(n-s, r_{n-s})}{1 - p_{n, r_n}(0)}, \quad n \geq 2. \quad (5)$$

If the sequence $\{r_n\}$ is known, then (5) can be used recursively to get the sequence $L(n, r_n)$ starting from $L(0, r_0) = L(1, r_0) = r_0$.

A recursive expression of $p_{n, r_n}(s, c)$, the probability of having s successes and c collided slots in the first frame, was given in [7]. To provide analytical evaluations of (5) we need, however, a closed-form expression for $p_{n, r_n}(s)$. This is given in the following

Theorem 1: The distribution $p_{n, r}(i)$ is given by

$$p_{n, r}(i) = \sum_{k=i}^m (-1)^{k+i} \binom{k}{i} X_{n, r}(k), \quad 0 \leq i \leq m, \quad (6)$$

where $m = \min\{n, r\}$, and

$$X_{n, r}(k) = \binom{r}{k} \frac{n!}{(n-k)!} \left(\frac{1}{r}\right)^k \left(\frac{r-k}{r}\right)^{n-k}, \quad (7)$$

with $k \leq m$.

Furthermore we have

$$p_{n+1, r+1}(i+1) = p_{n, r}(i) \frac{n+1}{i+1} \left(\frac{r}{r+1}\right)^n, \quad 0 \leq i \leq m. \quad (8)$$

Proof: Let A_1, A_2, \dots, A_r be r non-disjoint events. The probability that exactly t among these events jointly occur is given by [11]

$$P_t = X_t - \binom{t+1}{t} X_{t+1} + \binom{t+2}{t} X_{t+2} - \dots \pm \binom{r}{t} X_r \quad (9)$$

where

$$\begin{aligned}
X_1 &= \sum P(A_i) \\
X_2 &= \sum_{i \neq j} P(A_i A_j) \\
X_3 &= \sum_{i \neq j \neq k} P(A_i A_j A_k) \\
&\dots \dots \dots
\end{aligned} \tag{10}$$

and so on. Summations involve all possible combinations in such a way that each n -string appears just once, and the number of the terms X_k is $\binom{r}{k}$.

In our case the event A_i is defined as the occurrence of just one transmission, out of n , in slot i of a frame composed of r slots, and the probability of any of the k -string is given by

$$P(A_{j_1} A_{j_2} \dots A_{j_k}) = \frac{n!}{(n-k)!} \left(\frac{1}{r}\right)^k \left(\frac{r-k}{r}\right)^{n-k},$$

$k \leq m$, which by (9) and (10) proves the first part of the theorem.

The proof of the second part comes from (6) and (7), observing that

$$X_{n+1, r+1}(k+1) = \frac{n+1}{k+1} \left(\frac{r}{r+1}\right)^n X_{n, r}(k).$$

and rearranging terms. ■

Let now $L_o(n)$ represent the minimum average time to solve n tags over all sequences $\{r_i\}$, $0 \leq i \leq n$. Clearly (5) can be written as

$$L_o(n) = \frac{r_o + \sum_{s=1}^m p_{n, r_o}(s) L_o(n-s)}{1 - p_{n, r_o}(0)}, \quad n \geq 2. \tag{11}$$

Later in the paper we show that $r_o = n$ and therefore, by recursion, that setting $r_n = n$ at all frames is the optimal strategy.

In the remainder of the paper, for the sake of compactness, when $r = n$ we use a single subscript in the notation, e.g. \mathcal{S}_n in place of the more general $\mathcal{S}_{n, n}$.

In the analysis that follows we make use of some properties of R.V. \mathcal{S} , that are grouped in the following list

Properties A

- 1) $E[\mathcal{S}_{n, r}] = S_{n, r} = n \left(1 - \frac{1}{r}\right)^{n-1}$.
- 2) $S_{n, r}$ is maximized for $r = n$, i.e. $S_{n, r} \leq S_n$ for any pair (n, r) .
- 3) The normalized average S_n/n is a decreasing function of n such that $\lim_{n \rightarrow \infty} \frac{S_n}{n} = e^{-1}$.
- 4) $S_n - S_{n+1} < (1 - e^{-1})$, for all $n \geq 1$.
- 5) $\text{VAR}[\mathcal{S}_{n, r}] = S_{n, r} + S_{n, r} S_{n-1, r-1} - S_{n, r}^2$.
- 6) The normalized variance $\text{VAR}[\mathcal{S}_n]/n$ is a decreasing function of n such that $\lim_{n \rightarrow \infty} \frac{\text{VAR}[\mathcal{S}_n]}{n} = e^{-1} - e^{-2}$.
- 7) $\text{VAR}[\mathcal{S}_{n+1}] - \text{VAR}[\mathcal{S}_n] \leq e^{-1} - e^{-2}$, for all $n \geq 1$.

Properties 1 to 3 are well known and are here restated just for reference. Property 5 is proved in Lemma 2. Properties 4, 6 and 7 are easily proved with standard analytical tools starting from closed-form expressions. Other properties that are used throughout the paper can also be proved with standard tools and are listed next. The latter are referred to R.V. $\mathcal{R}_{n,r} = n - \mathcal{S}_{n,r}$ and its moments $R_{n,r} = E[\mathcal{R}_{n,r}]$.

Properties B

- 1) R_n/n is an increasing function of n such that $\frac{R_n}{n} = 1 - e^{-1} - \frac{e^{-1}}{2n} - \frac{7e^{-1}}{24n^2} - \frac{3e^{-1}}{16n^3} + \mathcal{O}(n^{-4})$.
- 2) $R_{n+1} - R_n$ is a decreasing function of n with $(R_{n+1} - R_n) = 1 - e^{-1} + \frac{7e^{-1}}{24n^2} + \mathcal{O}(n^{-3})$.
- 3) $R_{n,n} - R_{n,n+1} < e^{-1}$, for all $n \geq 1$.

Let now call "selected strategy" the one that assumes $r_n = n$ at all frames. Expressions (4) and (11), with this strategy can be rewritten as follows-the indexing in L is omitted,

$$L(n) = n + \sum_{i=1}^n \pi_{n,n}(i)L(i), \quad n \geq 2 \quad (12)$$

$$L(n) = \frac{n + \sum_{i=1}^{n-1} \pi_{n,n}(i)L(i)}{1 - \pi_{n,n}(n)}, \quad n \geq 2, \quad (13)$$

where $\pi_{n,r}(i) = p_{n,r}(n-i)$ is the probability that i terminals, out of the initial n , remain to be transmitted at the end of the frame. Note that $\pi_{n,r}(1) = 0$ for any pair (n, r) , therefore the summation in (12) can be started from $i = 2$.

We now prove the following:

Theorem 2: with the selected strategy the efficiency in identifying n terminals is a decreasing function of n , i.e. the following inequalities hold for $n \geq 1$:

$$\frac{n+1}{L(n+1)} < \frac{n}{L(n)}. \quad (14)$$

Furthermore we have

$$L(n) < ne, \quad (15)$$

and

$$\lim_{n \rightarrow \infty} \frac{n}{L(n)} = e^{-1}. \quad (16)$$

Proof: for $n = 1$ inequality (14) holds because $L(1) = 1$ and $L(2) = 4$. We assume that $L(i)/i > L(i-1)/(i-1)$ for $i \leq n$ and show that it holds also for $i = n+1$. From (12) we have

$$\frac{L(n)}{n} = 1 + \frac{1}{n} \sum_{i=2}^n \pi_n(i)L(i). \quad (17)$$

Using R_n , the average backlog at the end of the first frame, we can write

$$\frac{L(n)}{n} = 1 + \frac{R_n}{n} \sum_{i=2}^n \pi_n(i) \frac{i}{R_n} \frac{L(i)}{i} \quad (18)$$

and setting

$$\gamma_n(i) = \pi_n(i) \frac{i}{R_n}, \quad (19)$$

(18) becomes

$$\frac{L(n)}{n} = 1 + \frac{R_n}{n} \sum_{i=2}^n \gamma_n(i) \frac{L(i)}{i}. \quad (20)$$

Using (20) with $n+1$ and solving for $L(n+1)$ yields

$$\frac{L(n+1)}{n+1} = \frac{1}{1 - \pi_{n+1}(n+1)} + \frac{R_{n+1}}{n+1} \sum_{i=2}^n \gamma_{n+1}^*(i) \frac{L(i)}{i}, \quad (21)$$

where we have set

$$\gamma_n^*(i) = \frac{\gamma_n(i)}{1 - \pi_n(n)}, \quad 0 \leq i \leq n-1. \quad (22)$$

By Property B1 R_n/n is an increasing function of n and we get

$$\frac{R_{n+1}}{n+1} - \frac{R_n}{n} > 0. \quad (23)$$

This inequality used into (21) yields

$$\frac{L(n+1)}{n+1} > \frac{1}{1 - \pi_{n+1}(n+1)} + \frac{R_n}{n} \sum_{i=2}^n \gamma_{n+1}^*(i) \frac{L(i)}{i} \geq 1 + \frac{R_n}{n} \sum_{i=2}^n \gamma_{n+1}^*(i) \frac{L(i)}{i}, \quad (24)$$

which, together with (20), finally provides

$$\frac{L(n+1)}{n+1} - \frac{L(n)}{n} > \frac{R_n}{n} \sum_{i=2}^n (\gamma_{n+1}^*(i) - \gamma_n(i)) \frac{L(i)}{i}. \quad (25)$$

By Lemma 5, γ_{n+1}^* is "shifted to the right" (for the definition see the Appendix) with respect to γ_n and $\{L(i)/i\}$ is increasing by the initial assumption. Therefore, by Lemma 1 the first part of the theorem is proved.

For the second part we assume that $L(i) < ie$ for $i \leq n$ and show that it holds also for $i = n+1$ (it is trivially $L(1) = 1 < e$). The assumption allows to write

$$\begin{aligned} L(n+1) &= \frac{n+1 + \sum_{i=2}^n \pi_{n+1}(i)L(i)}{1 - \pi_{n+1}(n+1)} < \frac{n+1 + \sum_{i=2}^n \pi_{n+1}(i)ie}{1 - \pi_{n+1}(n+1)} \\ &< \frac{n+1 + (R_{n+1} - (n+1)\pi_{n+1}(n+1))e}{1 - \pi_{n+1}(n+1)}. \end{aligned} \quad (26)$$

Again, by Property B1, we have

$$R_n < n(1 - e^{-1}), \quad (27)$$

which, used in (26), finally provides

$$L(n+1) < \frac{(n+1 - (n+1) \cdot \pi_{n+1}(n+1))e}{1 - \pi_{n+1}(n+1)} = (n+1)e. \quad (28)$$

For the third part we use equation (20) observing that by Lemma 7 we have for all i

$$\lim_{n \rightarrow \infty} \gamma_n(i) = 0, \quad (29)$$

and therefore we can use the Toeplitz theorem [12, theorem 2]

$$\lim_{n \rightarrow \infty} \sum_{i=2}^n \gamma_n(i) \frac{L(i)}{i} = \lim_{n \rightarrow \infty} \frac{L(n)}{n} = X. \quad (30)$$

Also, from Property B1 we have

$$\lim_{n \rightarrow \infty} \frac{R_n}{n} = 1 - e^{-1},$$

then we can write (20) in the limit as

$$X = 1 + (1 - e^{-1})X \quad (31)$$

that solved for X^{-1} provides result (16). ■

Theorem 2 shows that the average IP length with the selected strategy can be written as

$$L(n) = ne - \varepsilon(n), \quad (32)$$

$\varepsilon(n) > 0$, and that

$$\lim_{n \rightarrow \infty} \frac{\varepsilon(n)}{n} = 0. \quad (33)$$

Denoting

$$\Delta\varepsilon(n+1) = \varepsilon(n+1) - \varepsilon(n)$$

we further have:

Theorem 3: The following inequality holds

$$\Delta\varepsilon(n) > \frac{\lambda}{n-2}, \quad n \geq n_0, \quad (34)$$

with $n_0 = 221$ and $\lambda = 1.08$.

Proof: by equation (12) we have

$$\begin{aligned} \varepsilon(n) &= ne - L_n = ne - n - \sum_{i=2}^n \pi_n(i)(ie - \varepsilon(i)) \\ &= n(e-1) - R_n e + \sum_{i=2}^n \pi_n(i)\varepsilon(i), \quad n \geq 2, \end{aligned}$$

and

$$\Delta\varepsilon(n+1) = e - 1 - (R_{n+1} - R_n)e + \sum_{i=2}^{n+1} (\pi_{n+1}(i) - \pi_n(i))\varepsilon(i), \quad n \geq 2, \quad (35)$$

having assumed $\pi_n(n+1) = 0$. By Property B2 we can write

$$R_{n+1} - R_n = (1 - e^{-1}) + z_{n+1}, \quad (36)$$

the error increase becomes

$$\Delta\varepsilon(n+1) = -z_{n+1}e + \sum_{i=2}^{n+1} (\pi_{n+1}(i) - \pi_n(i))\varepsilon(i), \quad n \geq 2. \quad (37)$$

By setting $\Delta\varepsilon(1) = \varepsilon(1)$, expression above can be written as

$$\Delta\varepsilon(n+1) = -z_{n+1}e + \sum_{i=2}^{n+1} (\pi_{n+1}(i) - \pi_n(i)) \sum_{j=1}^i \Delta\varepsilon(j), \quad n \geq 2, \quad (38)$$

and by inverting the order of summations we further have

$$\begin{aligned}\Delta\varepsilon(n+1) &= -z_{n+1}e + \sum_{j=1}^{n+1} \Delta\varepsilon(j) \left(\sum_{i=j}^{n+1} (\pi_{n+1}(i) - \pi_n(i)) \right) \\ &= -z_{n+1}e + \sum_{i=1}^{n+1} \Delta\varepsilon(i) (P(\mathcal{R}_{n+1} \geq i) - P(\mathcal{R}_n \geq i)), \quad n \geq 2,\end{aligned}$$

where $P(\mathcal{R}_{n+1} \geq n+1) = \pi_{n+1}(n+1)$ and $P(\mathcal{R}_n \geq n+1) = 0$. Solving for $\Delta\varepsilon(n+1)$, we get

$$\Delta\varepsilon(n+1) = \frac{-z_{n+1}e + \sum_{i=1}^n \Delta\varepsilon(i) (P(\mathcal{R}_{n+1} \geq i) - P(\mathcal{R}_n \geq i))}{1 - \pi_{n+1}(n+1)}, \quad n \geq 2. \quad (39)$$

Being

$$\sum_{i=1}^n P(\mathcal{R}_n \geq i) = E[\mathcal{R}_n] = R_n, \quad (40)$$

by Lemma 8 we have distribution α_n , with

$$\alpha_n(i) = \frac{P(\mathcal{R}_{n+1} \geq i) - P(\mathcal{R}_n \geq i)}{R_{n+1} - R_n - \pi_{n+1}(n+1)}, \quad 1 \leq i \leq n, \quad (41)$$

$\sum_{i=1}^n \alpha_n(i) = 1$, which, used in (39) with (36) yields

$$\Delta\varepsilon(n+1) = \frac{-z_{n+1}e}{1 - \pi_{n+1}(n+1)} + \frac{(1 - e^{-1}) + z_{n+1} - \pi_{n+1}(n+1)}{1 - \pi_{n+1}(n+1)} \sum_{i=1}^n \Delta\varepsilon(i) \alpha_n(i), \quad n \geq 2. \quad (42)$$

In order to lower bound the right hand side above we observe that, by Theorem 2, the summation is always positive, and that by Lemma 11, and (36), for $n \geq 49$ the following inequality holds

$$z_{n+1} - \pi_{n+1}(n+1) > 0.$$

Therefore, for $n \geq 49$, (42) together with inequality $\pi_{n+1}(n+1) > 0$ provides

$$\Delta\varepsilon(n+1) > \frac{-z_{n+1}e}{1 - \pi_{n+1}(n+1)} + (1 - e^{-1}) \sum_{i=1}^n \Delta\varepsilon(i) \alpha_n(i), \quad n \geq 2. \quad (43)$$

Let now define

$$g(x) = \frac{\lambda}{x-2}, \quad x \geq n_0 \quad (44)$$

$$g(x) = \frac{\lambda}{(n_0-2)^2} (2n_0 - 2 - x), \quad 1 \leq x < n_0 \quad (45)$$

where $g(x)$ in $1 \leq x < n_0$ assures that the whole $g(x)$ is convex in $x \geq 1$.

We can numerically verify that, for $\lambda = 1.08$, $n_0 = 221$, $n_1 = 442$, the following inequality holds

$$\Delta\varepsilon(\bar{n}) > \frac{\lambda}{\bar{n}-2} = g(\bar{n}), \quad n_0 \leq \bar{n} < n_1. \quad (46)$$

In the following, we assume that inequality (46) is verified for $\bar{n} \in [n_0, n]$, with $n \geq n_1$, and show that it is also satisfied for $\bar{n} \in [n_0, n+1]$, proving the theorem by induction.

The assumption above implies

$$\sum_{i=n_0}^n \Delta\varepsilon(i) \alpha_n(i) > \sum_{i=n_0}^n g(i) \alpha_n(i), \quad (47)$$

that used in (43) provides the the following inequality

$$\Delta\varepsilon(n+1) > -\frac{z_{n+1}e}{1-\pi_{n+1}(n+1)} + (1-e^{-1}) \left(V_n + \sum_{i=n_1}^n g(i)\alpha_n(i) \right), \quad (48)$$

where

$$V_n = \sum_{i=1}^{n_1-1} \Delta\varepsilon(i)\alpha_n(i). \quad (49)$$

Being $g(x)$ convex, we can apply Jensen's inequality:

$$\sum_{i=1}^n g(i)\alpha_n(i) > g\left(\sum_{i=1}^n i\alpha_n(i)\right) = g(J_n), \quad (50)$$

where we have denoted by J_n the average of distribution α_n . Inequality (50) can be rewritten as

$$\sum_{i=n_1}^n g(i)\alpha_n(i) > g(J_n) - W_n, \quad (51)$$

having defined

$$W_n = \sum_{i=1}^{n_1-1} g(i)\alpha_n(i). \quad (52)$$

Replacing (51) into (48) yields,

$$\Delta\varepsilon(n+1) > -\frac{z_{n+1}e}{1-\pi_{n+1}(n+1)} + (1-e^{-1})(V_n + g(J_n) - W_n). \quad (53)$$

Now we show that

$$C_n(n_1, \lambda) = V_n - W_n = \sum_{i=1}^{n_1-1} (\Delta\varepsilon(i) - g(i))\alpha_n(i) > 0, \quad (54)$$

and, therefore, can be disregarded in (53). Unfortunately terms $\Delta\varepsilon(i) - g(i)$ in (54) are not always positive for $i < n_0$, but we can numerically verify that for the given value of n_1 we have

$$\sum_{i=1}^{n_1-1} (\Delta\varepsilon(i) - g(i)) > 0. \quad (55)$$

Since we have $n_0 < (n_1 + 1)/2$, we can apply Lemma 9, that assures that the sequence $\{\alpha_n(i)\}_i$, $n \geq n_1$, is positive and increasing for $i \leq n_1$, and, therefore, taking into account (55), inequality (54) holds for any $n \geq n_1$.

Relation (53) now becomes

$$\Delta\varepsilon(n+1) > -\frac{z_{n+1}e}{1-\pi_{n+1}(n+1)} + (1-e^{-1})g(J_n). \quad (56)$$

By Lemma 12, valid for $n \geq 49$, we can exploit the following inequality

$$J_n \leq (1-e^{-1})(n+1) + e^{-1}, \quad (57)$$

in order to explicit $g(J_n)$. We use definition (46) for $g(i)$ since we can numerically verify that, being $n \geq n_1$, we have $J_n > n_0$. The substitution yields

$$\Delta\varepsilon(n+1) > -\frac{z_{n+1}e}{1-\pi_{n+1}(n+1)} + \frac{\lambda(1-e^{-1})}{(1-e^{-1})(n+1) + e^{-1} - 2}. \quad (58)$$

Inequality (34) is verified for $n + 1$ if

$$-\frac{z_{n+1}e}{1 - \pi_{n+1}(n+1)} + \frac{\lambda(1 - e^{-1})}{(1 - e^{-1})(n+1) + e^{-1} - 2} > \frac{\lambda}{n-1} \quad (59)$$

or

$$\frac{\lambda}{n-1/(1 - e^{-1})} - \frac{\lambda}{n-1} > \frac{z_{n+1}e}{1 - \pi_{n+1}(n+1)}. \quad (60)$$

The left-hand side of the above inequality is greater than zero. Since by Property B2 z_{n+1} tends to zero as n^{-2} , the above inequality is definitely verified for large n , say $n > n_2$. Specifically, for $\lambda = 1.08$ we have $n_2 = 2 < n_1$, thus proving the theorem.

We note that the bound can be strengthened, i.e., λ can be increased beyond 1.08, by increasing n_1 , if we are able to show numerically that the range of validity of inequality (46), $n_0 \leq i \leq 2n_0$, can be extended by increasing n_0 . However, we expect that increasing λ beyond some λ^* makes (54) no longer true for any n_0 . More on this limit λ^* is discussed later on. ■

Corollary 1: error $\varepsilon(n)$ is an increasing function of n for $n \geq 2$.

Proof: Theorem 3 shows that $\varepsilon(n)$ is an increasing function of n for $n \geq 221$. However, it can be assessed numerically, by (13), that it is increasing also in $2 \leq n \leq 221$. ■

Corollary 2: error ε is lower bounded by

$$\varepsilon(n) > 0.4562 + 1.08 \ln(n-1), \quad n \geq 221. \quad (61)$$

Proof: in fact, we can write

$$\varepsilon(n) > \varepsilon(n_0) + \sum_{i=n_0+1}^n \frac{\lambda}{i-2} > \varepsilon(n_0) + \int_{n_0+1}^{n+1} \frac{\lambda}{x-2} dx = \varepsilon(n_0) + \lambda \ln \frac{n-1}{n_0-1} = H + \lambda \ln(n-1), \quad (62)$$

from which (61) is derived by adopting the values of λ , n_0 , and $\varepsilon(n_0)$ of Theorem 3. ■

The next theorem provides an upper bound to the error.

Theorem 4: the following inequality holds:

$$\varepsilon(n) < \zeta \ln(n) - \frac{1}{n} + K, \quad n \geq 2, \quad (63)$$

being

$$\zeta = -\frac{0.5}{\ln(1 - e^{-1})} = 1.0900..., \quad (64)$$

and $K = 1.19$.

Proof: by (32) we have

$$\begin{aligned} \varepsilon(n) &= ne - L(n) = ne - n - \sum_2^n \pi_n(i)(ie - \varepsilon(i)) \\ &= ne - n - R_n e + \sum_2^n \pi_n(i)\varepsilon(i). \end{aligned} \quad (65)$$

Using the expression (Property B1)

$$\frac{R_n}{n} = 1 - e^{-1} - \xi_n$$

(65) becomes

$$\varepsilon(n) = n(e - 1) - n((1 - e^{-1}) - \xi_n)e + \sum_2^n \pi_n(i)\varepsilon(i) = ne\xi_n + \sum_2^n \pi_n(i)\varepsilon(i),$$

and, solving for $\varepsilon(n)$, we get

$$\varepsilon(n) = \frac{ne\xi_n + \sum_2^{n-1} \pi_n(i)\varepsilon(i)}{1 - \pi_n(n)}. \quad (66)$$

Let us now define

$$h(x) = \zeta \ln(x) - \frac{1}{x} + K, \quad x \geq 2 \quad (67)$$

$$h(x) = h'(x), \quad 0 \leq x < 2 \quad (68)$$

Where $h'(x) \geq 0$ is chosen in such a way that the whole function $h(x)$ is concave in $x \geq 0$, which is possible if $K = 1.19$.

We can numerically verify that, for $K = 1.19$ and $n = 2$, the following inequality holds

$$\varepsilon(n) < \zeta \ln(n) - \frac{1}{n} + K = h(n). \quad (69)$$

In the following, we assume that inequality (69) is verified for $n - 1 \geq 2$, and show that it is also satisfied for n , proving the theorem by induction. This assumption applied to (66), implies

$$\varepsilon(n) < \frac{ne\xi_n + \sum_2^{n-1} \pi_n(i)h(i)}{1 - \pi_n(n)}. \quad (70)$$

Owing to the concavity of $h(x)$, we can use Jensen's inequality as follows:

$$\sum_2^n \pi_n(i)h(i) \leq \sum_0^n \pi_n(i)h(i) \leq h(R_n), \quad (71)$$

rewritten as

$$\sum_2^{n-1} \pi_n(i)h(i) < h(R_n) - \pi_n(n)h(n). \quad (72)$$

Substituting (72) into (70) yields

$$\varepsilon(n) < \frac{ne\xi_n + h(R_n) - \pi_n(n)h(n)}{1 - \pi_n(n)}, \quad (73)$$

which, expliciting $h(i)$, and using (Property B1)

$$R_n \leq n(1 - e^{-1}), \quad (74)$$

becomes,

$$\varepsilon(n) < \frac{ne\xi_n + \zeta \ln(n(1 - e^{-1})) - \frac{1}{n(1 - e^{-1})} + K - \pi_n(n)(\zeta \ln(n) - \frac{1}{n} + K)}{1 - \pi_n(n)}. \quad (75)$$

After a rearrangement of terms we get

$$\varepsilon(n) < \zeta \ln(n) - \frac{1}{n} + K + \frac{ne\xi_n + \zeta \ln(1 - e^{-1}) - \frac{e^{-1}}{n(1 - e^{-1})}}{1 - \pi_n(n)}. \quad (76)$$

By Property B1, $ne\xi_n$ is a decreasing function with limit 0.5. Therefore, the last term in (76) is negative for $n = 2$ and for any $n \geq 2$, iff $\zeta \leq -0.5/\ln(1 - e^{-1})$, a condition met by definition (64). Therefore, (69) is verified for any such n , and the theorem is proved. ■

Theorem 5: the average IP length, $L(n)$, is such that

$$L(n) \sim ne - \zeta \ln(n)$$

Proof: owing to (32), we must prove that

$$\lim_{n \rightarrow \infty} \frac{\varepsilon(n)}{\zeta \ln(n)} = 1. \quad (77)$$

As shown in the previous theorem, the error can be rewritten as

$$\varepsilon(n) = \frac{ne\xi_n + \sum_{i=2}^{n-1} \pi_n(i)\varepsilon(i)}{1 - \pi_n(n)},$$

and taking limit (77) leads to

$$l = \lim_{n \rightarrow \infty} \frac{\varepsilon(n)}{\zeta \ln(n)} = \lim_{n \rightarrow \infty} \frac{1}{\zeta} \sum_{i=2}^{n-1} \frac{\pi_n(i)\varepsilon(i)}{\ln(n)}. \quad (78)$$

From Theorem 4 and Corollary 2 we know that the error takes the form

$$\varepsilon(n) = \zeta \ln(n) + \mu(n), \quad n \geq 2,$$

where $\mu(n)$ is a bounded function. Substituting for $\varepsilon(i)$ into (78) gives

$$l = \lim_{n \rightarrow \infty} \sum_{i=2}^{n-1} \frac{\pi_n(i) \ln(i)}{\ln(n)} + \lim_{n \rightarrow \infty} \frac{1}{\zeta} \sum_{i=2}^{n-1} \frac{\pi_n(i) \mu(i)}{\ln(n)} = \lim_{n \rightarrow \infty} \sum_{i=2}^{n-1} \frac{\pi_n(i) \ln(i)}{\ln(n)}, \quad (79)$$

having exploited the fact that $\sum_{i=2}^{n-1} \pi_n(i) \mu(i)$ is also bounded.

Let now consider the following transformation of random variable \mathcal{R}_n :

$$\mathcal{X}_n = \frac{\mathcal{R}_n}{R_n} + 1,$$

that is defined in

$$1 \leq \mathcal{X}_n \leq \frac{n}{R_n} + 1.$$

Using this transformation in (79) provides

$$\begin{aligned} l &= \lim_{n \rightarrow \infty} \sum_{x_i=x_2}^{x_{n-1}} \frac{P(\mathcal{X}_n = x_i) \ln(x_i R_n - 1)}{\ln(n)} = \\ &= \lim_{n \rightarrow \infty} \sum_{x_i=x_2}^{x_{n-1}} \frac{P(\mathcal{X}_n = x_i) \ln(R_n)}{\ln(n)} + \lim_{n \rightarrow \infty} \sum_{x_i=x_2}^{x_{n-1}} \frac{P(\mathcal{X}_n = x_i) \ln(x_i - 1/R_n)}{\ln(n)} = \\ &= l_1 + l_2, \end{aligned}$$

where, taking into account the range of \mathcal{X}_n , the second limit l_2 can be bounded as

$$0 = \lim_{n \rightarrow \infty} \frac{\ln(1 - 1/R_n)}{\ln(n)} \leq l_2 \leq \lim_{n \rightarrow \infty} \frac{\ln((n-1)/R_n + 1)}{\ln(n)} = 0,$$

and therefore $l_2 = 0$. Finally, from Property B1 it follows that

$$l = l_1 = \lim_{n \rightarrow \infty} \sum_{x_i = x_2}^{x_{n-1}} \frac{P(\mathcal{X}_n = x_i) \ln(n(1 - e^{-1}))}{\ln(n)} = 1,$$

that proves (77) and the theorem. ■

We now are in the position to prove the main theorem of this paper:

Theorem 6: $L(n, r)$ is minimized by the strategy that at each frame sets the frame length r equal to the backlog size n .

Proof: equation (5) can be conveniently rewritten as

$$L(n, r) = \frac{r}{1 - \pi_{n,r}(n)} + \frac{\sum_{i=2}^{n-1} \pi_{n,r}(i) L(i, r_i)}{1 - \pi_{n,r}(n)} = l_1 + l_2, \quad n \geq 2, \quad (80)$$

where l_1/r represents the average number of frames needed to have the first success. Denoting by S^* the average number of successes in l_1

$$S^* = \frac{\sum_{s=1}^m p_{n,r}(s)s}{1 - p_{n,r}(0)},$$

and by $R^* = n - S^*$ the remaining backlog, we write the efficiency as follows

$$\eta(n, r) = \frac{n}{L} = \frac{S^* + R^*}{L} = \frac{S^*}{l_1} \frac{l_1}{L} + \frac{R^*}{l_2} \frac{l_2}{L} = \eta_1 \frac{l_1}{L} + \eta_2 \frac{l_2}{L}. \quad (81)$$

We note that

$$\eta_1 = \frac{S^*}{l_1} = \frac{S}{r} = \frac{n}{r} \left(1 - \frac{1}{r}\right)^{n-1}, \quad (82)$$

which is maximized by setting $r = n$, and

$$\eta_2 = \frac{R^*}{l_2} = \frac{\sum_{i=2}^{n-1} \pi_{n,r}(i) i}{\sum_{i=2}^{n-1} \pi_{n,r}(i) L(i, r_i)}.$$

We now assume that the strategy $r_i = i$ is applied in the expression above, and show that $\eta(n, r)$ is maximized by setting $r = n$, as for η_1 . Since with $n = 2$ the optimal setting is easily proved to be $r = 2$, this proves the theorem by induction.

The proof is subdivided into two parts. In the first part we show that η decreases when changing from $r = n$ to $r = n - k$, with $1 \leq k \leq n - 2$. In the second part we show that η again decreases when changing from $r = n$ to $r = n + k$, with $k \geq 1$. In the first part we compare η, η_1 and η_2 in the two cases $r = n$ and $r = n - k$, that we refer to, in the indexing, as cases A and B respectively.

A. Part I

Owing to (82) we know that $\eta_{1A} > \eta_{1B}$. We now show that we have also $\eta_{2A} > \eta_{2B}$, therefore leading to $\eta_A > \eta_B$. We have

$$\eta_2 = \frac{R^*}{l_2} = \frac{\sum_{i=2}^{n-1} \pi_{n,r}(i)i}{\sum_{j=2}^{n-1} \pi_{n,r}(j)L(j)} = \sum_{i=2}^{n-1} \frac{i}{L(i)} \frac{\pi_{n,r}(i)L(i)}{\sum_{j=2}^{n-1} \pi_{n,r}(j)L(j)} = \sum_{i=2}^{n-1} \eta(i)\beta_{n,r}(i), \quad (83)$$

where

$$\eta(i) = \frac{i}{L(i)}, \quad i > 1, \quad \eta(1) = 1,$$

and

$$\beta_{n,r}(i) = \frac{\pi_{n,r}(i)L(i)}{\sum_{j=2}^{n-1} \pi_{n,r}(j)L(j)}$$

with

$$\sum_{i=2}^{n-1} \beta_{n,r}(i) = 1,$$

that is, η_2 is expressed as the average of $\eta(i)$ weighted with distribution $\beta_{n,r}$. The difference of the two cases is given by

$$\eta_{2B} - \eta_{2A} = \sum_{i=2}^{n-1} (\beta_{n,n-k}(i) - \beta_{n,n}(i))\eta(i) < 0. \quad (84)$$

The result comes from Lemma 1 by observing that the distribution $\beta_{n,n-k}$ is, by Lemma 6, "shifted to the right" with respect to $\beta_{n,n}$, and that, by Theorem 2, $\{\eta(i)\}$ is a decreasing sequence. This concludes the first part of the proof.

In the following we refer to the case $r = n + k$ as case C. Here, the same argument that has lead to (84) can be proven to show that $\eta_{2C} > \eta_{2A}$ while we still have $\eta_{1C} < \eta_{1A}$, therefore a new approach is needed.

B. Part II

This part is proven by showing that

$$\Delta l_{2C} = l_{2A} - l_{2C} < k \quad (85)$$

so that the gain in l_{2C} is not enough to compensate the increase of k slots in the first frame. Using (32) we have

$$L(n) = ne - \varepsilon(n), \quad \varepsilon(n) > 0. \quad (86)$$

We then have

$$\begin{aligned} \Delta l_{2C} &= \sum_{i=2}^{n-1} (\pi_{n,n}(i) - \pi_{n,n+k}(i))L(i) = \sum_{i=2}^{n-1} (\pi_{n,n}(i) - \pi_{n,n+k}(i))(ie - \varepsilon(i)) \\ &= (R_{n,n} - R_{n,n+k} - \pi_{n,n}(n) + \pi_{n,n+k}(n))e - \sum_{i=2}^{n-1} (\pi_{n,n}(i) - \pi_{n,n+k}(i))\varepsilon(i). \end{aligned} \quad (87)$$

By property B3 we know that

$$R_{n,n} - R_{n,n+k} < ke^{-1}, \quad (88)$$

and, by Lemma 4 we have also

$$\pi_{n,n}(n) > \pi_{n,n+k}(n). \quad (89)$$

Using both (88) and (89) in (87) provides

$$\Delta l_{2C} < k - \sum_{i=2}^{n-1} (\pi_{n,n}(i) - \pi_{n,n+k}(i)) \varepsilon(i). \quad (90)$$

By Corollary 2 $\{\varepsilon(i)\}$ is an increasing sequence for $i \geq 2$. Also, by Lemma 4 the distribution $\pi_A = \pi_{n,n}$ is "shifted to the right" with respect to $\pi_C = \pi_{n,n+k}$. Therefore, by Lemma 1 we have

$$\sum_{i=2}^{n-1} (\pi_{n,n}(i) - \pi_{n,n+k}(i)) \varepsilon(i) > 0, \quad (91)$$

which proves (85) and, finally, the theorem. \blacksquare

APPENDIX A

Considering two probability distributions π_A and π_B , we say that the latter is "shifted to the right" of the former when there is a value i_0 such that

$$\pi_A(i) \leq \pi_B(i), \quad i > i_0$$

and

$$\pi_A(i) \geq \pi_B(i), \quad i \leq i_0.$$

Lemma 1: if distribution π_B is shifted to the right with respect to distribution π_A , and $\{f(i)\}$ is a non-decreasing sequence, the following holds

$$\sum_i (\pi_B(i) - \pi_A(i)) f(i) \geq 0. \quad (92)$$

Proof: the limit case is when $\pi_B = \pi_A$, where (92) holds with the equality. One can obtain a generic distribution π_B shifted to the right with respect to distribution π_A by starting from a distribution $\pi'_B = \pi_A$ and dragging positive quantities $\Delta\pi(i)$ from the probabilities $\pi'_B(i \leq x)$, and drop them onto the bins $i > x$. From this construction it follows that

$$\begin{aligned} \sum_i \pi_B(i) f(i) &= \sum_{i \leq x} \pi_B(i) f(i) + \sum_{i > x} \pi_B(i) f(i) \\ &= \sum_{i \leq x} (\pi'_B(i) - \Delta\pi(i)) f(i) + \sum_{i > x} (\pi'_B(i) + \Delta\pi(i)) f(i) \\ &= \sum_i \pi'_B(i) f(i) - \left(\sum_{i \leq x} \Delta\pi(i) f(i) - \sum_{i > x} \Delta\pi(i) f(i) \right), \end{aligned} \quad (93)$$

where

$$\sum_{i \leq x} \Delta\pi(i) = \sum_{i > x} \Delta\pi(i).$$

Substituting (93) into (92) one gets

$$\begin{aligned} \sum_i (\pi_B(i) - \pi_A(i))f(i) &= \sum_i (\pi'_B(i) - \pi_A(i))f(i) - \left(\sum_{i \leq x} \Delta\pi(i)f(i) - \sum_{i > x} \Delta\pi(i)f(i) \right) \\ &= - \sum_{i \leq x} \Delta\pi(i)f(i) + \sum_{i > x} \Delta\pi(i)f(i), \end{aligned}$$

and since $\{f(i)\}$ is a non-decreasing sequence

$$- \sum_{i \leq x} \Delta\pi(i)f(i) + \sum_{i > x} \Delta\pi(i)f(i) \geq f(x) \left(\sum_{i \leq x} \Delta\pi(i) - \sum_{i > x} \Delta\pi(i) \right) = 0.$$

Hence it follows the thesis. ■

Lemma 2: the variance of the number of successes $\mathcal{S}_{n,r}$ can be expressed as

$$\text{VAR}[\mathcal{S}_{n,r}] = S_{n,r} + S_{n,r}S_{n-1,r-1} - S_{n,r}^2.$$

Proof: the number of successes $\mathcal{S}_{n,r}$ can be expressed as the sum of binary R.V. $\sigma_{n,r}(i) \in \{0, 1\}$ representing the success ($\sigma_{n,r}(i) = 1$) in slot i :

$$\mathcal{S}_{n,r} = \sum_{i=1}^r \sigma_{n,r}(i).$$

Its variance can be expressed as

$$\begin{aligned} \text{VAR}[\mathcal{S}_{n,r}] &= \sum_{i=1}^r \text{VAR}[\sigma_{n,r}(i)] + 2 \sum_{i=1}^r \sum_{j=1, j \neq i}^r \text{COVAR}[\sigma_{n,r}(i), \sigma_{n,r}(j)] = \\ &= r \text{VAR}[\sigma_{n,r}(i)] + r(r-1) \text{COVAR}[\sigma_{n,r}(i), \sigma_{n,r}(j)]. \end{aligned}$$

Since the probability of having a success in slot i coincides with $S_{n,r}/r$, we have

$$\text{VAR}[\sigma_i] = E[\sigma_i^2] - E[\sigma_i]^2 = P(\sigma_{n,r}(i) = 1) - P(\sigma_{n,r}(i) = 1)^2 = \frac{S_{n,r}}{r} - \left(\frac{S_{n,r}}{r} \right)^2.$$

Similarly we have

$$E[\sigma_{n,r}(i)\sigma_{n,r}(j)] = P(\sigma_{n,r}(i) = 1, \sigma_{n,r}(j) = 1) = \frac{S_{n,r}}{r} \cdot \frac{S_{n-1,r-1}}{r-1},$$

and

$$\text{COVAR}[\sigma_i, \sigma_j] = \frac{S_{n,r}}{r} \cdot \frac{S_{n-1,r-1}}{r-1} - \left(\frac{S_{n,r}}{r} \right)^2,$$

which proves the lemma. ■

Let now denote by $\pi_{n,r}$ the distribution of the back log size after the first frame, of length r with n transmissions. We obviously have

$$\pi_{n,r}(i) = p_{n,r}(n-i),$$

and the following holds

Lemma 3: distribution $\pi_{n+1,r+1}$ is shifted to right with respect to $\pi_{n,r}$.

Proof: the thesis implies that distribution $\{p_{n,r}(i)\}_i$ is shifted to right with respect to $\{p_{n+1,r+1}(i+1)\}_i$. By Theorem 1 we have

$$p_{n+1,r+1}(i+1) = p_{n,r}(i) \frac{n+1}{i+1} \left(\frac{r}{r+1} \right)^n$$

and, therefore,

$$p_{n+1,r+1}(i+1) - p_{n,r}(i) = p_{n,r}(i) \left(\frac{n+1}{i+1} \left(\frac{r}{r+1} \right)^n - 1 \right). \quad (94)$$

The thesis is proved because equation

$$\frac{n+1}{i+1} \left(\frac{r}{r+1} \right)^n - 1 = 0$$

presents only one real zero i_0 . ■

Lemma 4: distribution $\pi_{n,r}$ is shifted to right with respect $\pi_{n,r+1}$.

Proof: the thesis implies that distribution $p_{n,r+1}$ is shifted to right with respect $p_{n,r}$. Owing to (8) we have

$$p_{n,r+1}(i) = p_{n-1,r}(i-1) \frac{n}{i} \left(\frac{r}{r+1} \right)^{n-1} \quad (95)$$

and also

$$p_{n,r}(i) = p_{n-1,r-1}(i-1) \frac{n}{i} \left(\frac{r-1}{r} \right)^{n-1}, \quad (96)$$

and the ratio of previous relations is

$$\frac{p_{n,r+1}(i)}{p_{n,r}(i)} = \frac{p_{n-1,r}(i-1)}{p_{n-1,r-1}(i-1)} \left(\frac{r^2}{r^2-1} \right)^{n-1}. \quad (97)$$

Let now assume that the ratio

$$Y_{n,r}(i) = \frac{p_{n,r+1}(i)}{p_{n,r}(i)} \quad (98)$$

is a non-decreasing function of i and that

$$Y_{n,r}(i_0) \leq 1, \quad Y_{n,r}(i_0+1) \geq 1.$$

Then, owing to (97), $Y_{n+1,r+1}(i)$ is a non-decreasing function of i . Furthermore, condition $Y_{n+1,r+1}(i) > 1$ can not hold for all i because this would imply $p_{n+1,r+1}(i) > p_{n+1,r}(i)$ for all i , clearly impossible. Therefore, for some i_1 it must hold

$$Y_{n+1,r+1}(i_1) \leq 1, \quad Y_{n+1,r+1}(i_1+1) \geq 1.$$

This proves the thesis. ■

Lemma 5: distribution γ_{n+1}^* is shifted to right with respect γ_n .

Proof: $\gamma_n(i)$ is related to $\pi_n(i)$ by (19) and by Lemma 3 we know that π_{n+1} is shifted to right with respect π_n . The scaling with $R_{n+1} > R_n$ changes the intersection point i_0 of the two distributions into $i_1 > i_0$. Nevertheless $i_1 < n+1$ since $\pi_{n+1}(n+1)/R_{n+1} > \pi_n(n+1)/R_n = 0$. The further multiplication by i does not change the intersection point that remains i_1 , and therefore γ_{n+1} is shifted to right with respect γ_n .

Finally γ_{n+1}^* is upscaled with respect to γ_{n+1} and truncated to $i \leq n$. This moves the intersection point with γ_n to $i_2 < i_1$. We note that we necessarily have $i_2 < n$ because if it were $i_2 = n$, we would have $\gamma_n(i) \leq \gamma_{n+1}^*(i)$ for all i , where for some i the inequality becomes "strictly less". But this can not be since both functions must sum to one. ■

Lemma 6: distribution $\beta_{n,r}$ is shifted to right with respect $\beta_{n,r+1}$.

The proof proceed as in Lemma 5 , but starting from Lemma 4 by which $\pi_{n,n-1}$ is shifted to right with respect $\pi_{n,n}$.

Lemma 7: the following limit holds for all i

$$\lim_{n \rightarrow \infty} \gamma_n(i) = 0. \quad (99)$$

Proof: relation (8) can be rewritten as

$$\pi_{n+1}(i) = \pi_n(i) \frac{n+1}{n+1-i} \left(\frac{n}{n+1} \right)^n. \quad (100)$$

If (100) is repeatedly applied k time we obtain

$$\pi_{n+k}(i) = \pi_n(i) \prod_{j=1}^k \frac{n+j}{n+j-i} \left(\frac{n+j-1}{n+j} \right)^{n+j-1}, \quad (101)$$

and using the inequality

$$\left(\frac{n+j-1}{n+j} \right)^{n+j-1} \leq \frac{1}{2} \quad (102)$$

that holds for all $n \geq 1$ and $j \geq 1$, relation (101) becomes

$$\pi_{n+k}(i) \leq \pi_n(i) \left(\frac{1}{2} \right)^k \prod_{j=1}^k \frac{n+j}{n+j-i} \quad (103)$$

$$< \pi_n(i) \left(\frac{1}{2} \right)^k \left(\frac{n+1}{n+1-i} \right)^k. \quad (104)$$

Since we are interested in the limit $n+k \rightarrow \infty$, for any i fixed in advance we can always chose n so that $\frac{n+1}{n+1-i}$ approaches 1 as close as we want. Say we choose

$$\frac{n+1}{n+1-i} = \frac{8}{10}, \quad (105)$$

then (104) becomes

$$\pi_{n+k}(i) < \pi_n(i) \left(\frac{4}{10} \right)^k, \quad (106)$$

and therefore

$$\lim_{k \rightarrow \infty} \pi_{n+k}(i) = 0. \quad (107)$$

We also have

$$\lim_{k \rightarrow \infty} \gamma_{n+k}(i) = \lim_{k \rightarrow \infty} \pi_{n+k}(i) \frac{i}{R_{n+k}} = 0 \quad (108)$$

since

$$\lim_{k \rightarrow \infty} \frac{i}{R_{n+k}} = 0. \quad (109)$$

Let now denote by R_n the V.C. representing the backlog size at the end of the first frame of length n with n transmissions. Then we have:

Lemma 8: vector $\{\alpha_n(i)\}_{i=0}^n$, with

$$\alpha_n(i) = \frac{P(\mathcal{R}_{n+1} \geq i) - P(\mathcal{R}_n \geq i)}{R_{n+1} - R_n - \pi_{n+1}(n+1)},$$

is a probability mass function, i.e.

$$\sum_{i=0}^n \alpha_n(i) = 1$$

and $\alpha_n(i) \geq 0$ for all $0 \leq i \leq n$.

Proof: we prove at first that

$$R_{n+1} - R_n - \pi_{n+1}(n+1) > 0$$

for all n . From property B2 the condition above is equivalent to

$$\pi_{n+1}(n+1) < 1 - e^{-1}.$$

Using Theorem 1 we can write

$$\begin{aligned} \pi_{n+1}(n+1) &= p_{n+1}(0) = 1 - \sum_{i=1}^{n+1} p_{n+1}(i) \\ &= 1 - (n+1) \left(\frac{n}{n+1} \right)^n \sum_{i=0}^n \frac{p_n(i)}{i+1}, \end{aligned}$$

and, thanks to the convexity of the function $(i+1)^{-1}$, Jensen's inequality leads to

$$\begin{aligned} \pi_{n+1}(n+1) &\leq 1 - (n+1) \left(\frac{n}{n+1} \right)^n \frac{1}{S_n + 1} \\ &\leq 1 - \left(\frac{n}{n+1} \right)^n < 1 - e^{-1}, \end{aligned}$$

where in the second inequality we used $S_n \leq n$.

Now we show that

$$\bar{\alpha}_n(i) = P(\mathcal{R}_{n+1} \geq i) - P(\mathcal{R}_n \geq i) \geq 0, \quad 0 \leq i \leq n+1. \quad (110)$$

For $i = n+1$ condition (110) is trivially verified, since it is $\bar{\alpha}_n(n+1) = 0$. By induction, assume that $\bar{\alpha}_n(i+1) \geq 0$.

From Lemma 3 we know that π_{n+1} is shifted to the right with respect to π_n . This means that there exists one index x such that $\pi_{n+1}(i) \geq \pi_n(i)$ for all $i > x$. This implies that for $i > x$

$$\begin{aligned} \bar{\alpha}_n(i) &= P(\mathcal{R}_{n+1} \geq i) - P(\mathcal{R}_n \geq i) = \pi_{n+1}(i) + P(\mathcal{R}_{n+1} \geq i+1) - \pi_n(i) - P(\mathcal{R}_n \geq i+1) \\ &= \bar{\alpha}_n(i+1) + \pi_{n+1}(i) - \pi_n(i) \geq \bar{\alpha}_n(i+1) \geq 0, \end{aligned}$$

thus proving that the terms $\bar{\alpha}_n(i)$, and hence $\alpha_n(i)$, are positive for $i > x$.

For $i \leq x$, since $\pi_{n+1}(i) \leq \pi_n(i)$, the sequence $\{\bar{\alpha}_n(i)\}$ cannot increase when i decreases. Moreover, the infimum value of this sequence is

$$\bar{\alpha}_n(0) = P(\mathcal{R}_{n+1} \geq 0) - P(\mathcal{R}_n \geq 0) = 1 - 1 = 0,$$

meaning that it must be $\bar{\alpha}_n(i) \geq 0$ for all $0 \leq i \leq n+1$, and, therefore, $\alpha_n(i) \geq 0$ for all $0 \leq i \leq n$.

Finally, we can write that

$$\begin{aligned} \sum_{i=0}^n \alpha_n(i) &= \frac{\sum_{i=0}^n \sum_{j=i}^n (\pi_{n+1}(j) - \pi_n(j))}{R_{n+1} - R_n} = \frac{\sum_{j=0}^n \sum_{i=0}^j (\pi_{n+1}(j) - \pi_n(j))}{R_{n+1} - R_n - \pi_{n+1}(n+1)} = \\ &= \frac{\sum_{j=0}^n (j+1)(\pi_{n+1}(j) - \pi_n(j))}{R_{n+1} - R_n - \pi_{n+1}(n+1)} = \frac{\sum_{j=0}^n j\pi_{n+1}(j) - \sum_{j=0}^n j\pi_n(j) + \sum_{j=0}^n \pi_{n+1}(j) - 1}{R_{n+1} - R_n - \pi_{n+1}(n+1)} = 1. \end{aligned}$$

■

Lemma 9: there exists an index $\bar{i}(n)$, $(n+1)/2 < \bar{i}(n) < (n+1)(1 - e^{-1})$ such that for any $i < \bar{i}(n)$

$$\alpha_n(i+1) - \alpha_n(i) > 0.$$

Proof: from the definition of α_n in Lemma 8 we see that

$$\alpha_n(i+1) - \alpha_n(i) \propto \pi_n(i) - \pi_{n+1}(i),$$

and, therefore, the sequence α_n increases if

$$\pi_n(i) - \pi_{n+1}(i) > 0.$$

Using (100), the previous inequality is rewritten as

$$1 - \frac{n+1}{n+1-i} \left(\frac{n}{n+1} \right)^n > 0,$$

that is verified for

$$i < (n+1) \left(1 - \left(\frac{n}{n+1} \right)^n \right) = \bar{i}(n).$$

Inequalities $(n+1)/2 < \bar{i}(n) < (n+1)(1 - e^{-1})$ can be verified by standard methods. ■

Lemma 10: the following inequality holds

$$\frac{\pi_{n+1}(i+1)}{\pi_{n+1}(i)} > \frac{\pi_n(i+1)}{\pi_n(i)}. \quad (111)$$

Proof: from (8) in the form (100) we have

$$\frac{\pi_{n+1}(i)}{\pi_n(i)} = \frac{n+1}{n+1-i} \left(\frac{n}{n+1} \right)^n < \frac{n+1}{n+1-(i+1)} \left(\frac{n}{n+1} \right)^n = \frac{\pi_{n+1}(i+1)}{\pi_n(i+1)},$$

which, proves the Lemma. ■

Lemma 11: the sequence $\{\pi_n(n)\}_n$ for $n > 16$ is bounded as:

$$\pi_n(n) < \bar{\pi}_n(n) = 3.47 \cdot 10^{-3} \cdot 0.9157^n + 59.79 \cdot 0.4157^n. \quad (112)$$

Furthermore, for $n \geq 49$ the following inequality holds

$$R_{n+1} - R_n - (1 - e^{-1}) > \pi_{n+1}(n+1). \quad (113)$$

Proof: from recursion (8) one has

$$\pi_n(n-1) = p_n(1) = p_{n-1}(0) \cdot S_n = \pi_{n-1}(n-1) \cdot S_n. \quad (114)$$

Once the number of terminals that participates in a frame is fixed, adding a slot to the frame decreases the probability of having no successes, or, in other terms, $\pi_{n+1}(n+1) < \pi_{n+1,n}(n)$. By using the recursive formula for computing distribution $p_{n+1,r}$ from distribution $p_{n,r}$, and using (114), one can write that

$$\begin{aligned} \pi_{n+1}(n+1) &< \pi_{n+1,n}(n) = \pi_n(n)P(\mathcal{X}=0) + \pi_n(n-1)P(\mathcal{X}=-1) \\ &= \pi_n(n)P(\mathcal{X}=0) + \pi_{n-1}(n-1)S_nP(\mathcal{X}=-1), \end{aligned} \quad (115)$$

where

$$P(\mathcal{X}=-1) = 1/n,$$

and

$$P(\mathcal{X}=0) = \frac{E[\mathcal{C}_n | S_n=0]}{n} < \frac{n}{2} \cdot \frac{1}{n} = \frac{1}{2}.$$

Substituting the two above expressions into (115), and using property A3, results

$$\pi_{n+1}(n+1) < \frac{1}{2}\pi_n(n) + \frac{S_{15}}{15}\pi_{n-1}(n-1),$$

for $n \geq 16$. This means that it is possible to build a sequence $\{\bar{\pi}_n(n)\}$, that upper bounds the true sequence $\{\pi_n(n)\}$, through the recurrence

$$\bar{\pi}_{n+1}(n+1) = 0.5 \bar{\pi}_n(n) + 0.381 \bar{\pi}_{n-1}(n-1),$$

for $n \geq 16$, with initial conditions $\bar{\pi}_{14}(14) = \pi_{14}(14) \approx 1.285 \cdot 10^{-3}$ and $\bar{\pi}_{15}(15) = \pi_{15}(15) \approx 8.106 \cdot 10^{-4}$. The solution of the above difference equation is

$$\bar{\pi}_n(n) = 3.47 \cdot 10^{-3} \cdot 0.9157^n + 59.79 \cdot (-0.4157)^n, \quad (116)$$

for $n \geq 16$. From this, bound (112) is immediate.

By property B2 the left-hand term of (113) decreases as $\mathcal{O}(n^{-2})$, whereas, by (112), the right-hand term decreases exponentially. Therefore, since we can numerically verify that (113) holds for $n = 49$, it holds for all $n > 49$. ■

Lemma 12: the average

$$J_n = \sum_{i=1}^n i\alpha_n(i) \quad (117)$$

is such that, for $n > 49$

$$J_n < (n+1)(1 - e^{-1}) + e^{-1}. \quad (118)$$

Proof: working on definition (117) yields

$$J_n = \frac{E[\mathcal{R}_{n+1}^2] - E[\mathcal{R}_n^2] - (2n+1)\pi_{n+1}(n+1)}{2(R_{n+1} - R_n - \pi_{n+1}(n+1))} + \frac{1}{2} \quad (119)$$

that further elaboration reduces to

$$J_n = \frac{R_{n+1} + R_n}{2} + \frac{\text{VAR}[\mathcal{R}_{n+1}] - \text{VAR}[\mathcal{R}_n] - (2n + 1 - (R_{n+1} + R_n))\pi_{n+1}(n + 1)}{2(R_{n+1} - R_n - \pi_{n+1}(n + 1))} + \frac{1}{2}. \quad (120)$$

Since it is $R_{n+1} + R_n \leq 2n + 1$, the term in parentheses can be omitted and we get

$$J_n < \frac{R_{n+1} + R_n}{2} + \frac{\text{VAR}[\mathcal{R}_{n+1}] - \text{VAR}[\mathcal{R}_n]}{2(R_{n+1} - R_n - \pi_{n+1}(n + 1))} + \frac{1}{2} \quad (121)$$

By Property A7 we have

$$\text{VAR}[\mathcal{R}_{n+1}] - \text{VAR}[\mathcal{R}_n] = \text{VAR}[\mathcal{S}_{n+1}] - \text{VAR}[\mathcal{S}_n] \leq e^{-1} - e^{-2} \quad (122)$$

and by Lemma 11 and for $n \geq 49$

$$R_{n+1} - R_n - \pi_{n+1}(n + 1) \geq R_{n+1} - R_n - \bar{\pi}_{n+1}(n + 1) > 1 - e^{-1}. \quad (123)$$

Also, from property B1 we have

$$\frac{R_{n+1} + R_n}{2} < \left(n + \frac{1}{2}\right) (1 - e^{-1}).$$

Using the preceding inequalities, (121) becomes

$$J_n < \left(n + \frac{1}{2}\right) (1 - e^{-1}) + \frac{e^{-1} - e^{-2}}{2(1 - e^{-1})} + \frac{1}{2} = (n + 1)(1 - e^{-1}) + e^{-1}, \quad (124)$$

for $n \geq 49$. ■

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